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# Phenomenological anharmonic vibrational models description for the ground state band energies of even-even nuclei

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**Abstract** : The results from the cubic polynomial (CP) formula of the square of the angular velocity and the nuclear moments of inertia are compared with those from the variable moment of inertia (VMI) model and the available experimental information on transition energies for yrast line in even-even nuclei. The evaluated model parameters lead to an excellent fit for all energy levels ($I \sim 24$) The calculated critical spin for backbending in the $\Im - \omega^2$ plot is found to be in agreement with the experimental data

**Keywords** : Nuclear structure, rare earth nuclei, moment of inertia, backbending

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## 1. Introduction

In the rare earth region, the energy levels of the double even nuclei usually manifest rotational band structures in the low spin region. As one goes to the higher spin region, the anomalous behaviour of the moment of inertia shows up [1]. There have been many microscopic theoretical models to study the rotational spectra in even-even nuclei. All of these are based on the extension of the Mottelson-Valation idea to Hartree-Fock Bogolivbov (HFB) theory [2]. On the phenomenological side, the variable moment of inertia model (VMI) seems to be one of the successful phenomenological models for even-even nuclei. This model have been extended by many authors. The extension for two and three parameters models, referred to as VMI12, VMI123 [3,4], are obtained by adding in the first order anharmonic term in the potential energy to the VMI model expression. The extension of the VMI model by Das and Banerjee [5] gives a four-parameter description (VMI1234 model). Mantri and Sood [6], demonstrated the equivalence of the mathematical formulations of the various models in respect of their energy expressions. In the present work, we have first tried to find an expression which would closely reproduce the energy levels of an anharmonic oscillator. A short comment has then been presented on the backbending behaviour which is predicted in our calculations specially in nuclei without any backbending or with weak backbending (Hf-Isotopes).

## 2. The model

Since we are concerned with the ground state bands only (the sequence $I^\pi = 0^+, 2^+, 4^+, 6^+, \ldots, etc$) in a vibrational picture, the total angular momentum $I$ can be written as $I = 2N$, where $N$ is the number of vibrational quanta. A phenomenological description of the energy spectra of an anhormonic oscillator can be written as

$$E(I) = aI + bI^2 + cI^3 + \cdots.$$

If this expression is truncated upto any finite number of terms, say, upto the terms involving $I^3$, the coefficient $a$, $b$ and $c$ would in general, have a $I$-dependence [7,8]. It is now interesting to derive the moments of inertia from the

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experimental data and to investigate how they change as the rotational angular velocity of the nucleus varies [9]. This angular velocity is defined by the canonical relation appropriate for an axial symmetric rotor.

The following expression for the moment of inertia is obtained

$$2\mathfrak{I}/\hbar^2 = \left[dE/dI(I+1)\right]^{-1}. \tag{1}$$

From the CP formula, we have the following expressions.

$$2\mathfrak{I}/\hbar^2 = (2I+1)\left[a+2bI+3cI^2+c\right]^{-1} \tag{2}$$

and
$$\hbar^2\ \omega^2 = \frac{I^2+I+1}{I^2+I+\frac{1}{4}}\left[a+2bI+3cI^2+c\right]^2. \tag{3}$$

## 3. Analysis of the data and discussion

We have made a least-square fit to the energy spectra to the ground state bands for all even-even nuclei. The model parameters $a$, $b$ and $c$ can be determined from fitting the three energy levels $E_2$, $E_4$ and $E_6$. We have determined the values of the parameters through least-square fitting of the first four to five energy levels. We have carried out a detailed analysis of the fits to the observed energy levels for 50 even-even nuclei $88 \le N \le 114$. The least square average fitted values of the parameters $a$, $b$ and $c$ in some studied nuclei are listed in Table 1.

**Table 1.** Least square fitted values of the parameters $a$, $b$ and $c$ for the anharmonic oscillator energy levels. The parameter values for the anharmonic oscillator are in arbitrary units (keV).

| Isotopes | $a$ | $b$ | $c$ |
|---|---|---|---|
| $^{158}$Dy | 13.650 | 19.053 | -0 6249 |
| $^{160}$Dy | 10.725 | 15.827 | -0.2558 |
| $^{162}$Er | 15 639 | 18.045 | -0 3579 |
| $^{164}$Er | 13 733 | 16.213 | -0 2407 |
| $^{168}$Yb | 12 063 | 16.021 | -0 2969 |

The quality of fits obtained by the CP formula is compared with that obtained by the variable moment of inertia (VMI) model and the experimental one for a set of representative nuclei. It can be seen (Tables 2–5) that the quality of fit, with an error around 1.14% between the calculated and experimental energy levels, is very good even upto $I = 26^+$ compared to those from Mantri and Sood [8]. The difference between eqs. (2, 3) and the same in Ref. [8] are apparent. This difference does not affect so much on the value and position of the backbending which is predicted in the calculations (Tables 2–5) but gives better fits with the experimental data.

The moment of inertia anomalies can be illustrated most clearly in the conventional plot of $2\mathfrak{I}/\hbar^2$ *versus* the square of rotational frequency $(\hbar\omega)^2$ (Figures 1, 2). A smooth increase of the moment of inertia with the angular velocity is observed for Hf isotopes (experimental results) and a

**Table 2.** Comparison of the present work with experimental rotational energies and the predictions of VMI model of (a) $^{158}$Dy [13], (b) $^{160}$Dy [
The units of $E$, $\mathfrak{I}$ and $\omega^2$ are keV, (MeV)$^{-1}$ and (MeV)$^2$ respectively

| $I$ | | $4^+$ | $6^+$ | $8^+$ | $10^+$ | $12^+$ | $14^+$ | $16^+$ | $18^+$ | $20^+$ | $22^+$ | $24^+$ | $26^+$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | (a) | | | | | | | |
| $E_{exp}$ | 98.94 | 317 26 | 637.88 | 1044.02 | 1519 9 | 2049.2 | 2612 5 | 3190 7 | 3781 5 | 4407.3 | | | |
| $E_c$ | 99 44 | 315.04 | 630.72 | 1030 4 | 1498.0 | 2017.44 | 2572 6 | 3147.52 | 3726 | 4292 | 4829 42 | 5322.22 | – |
| $E_{VMI}$ | 98.67 | 316 46 | 634 07 | 1034 75 | 1505 86 | 2038 01 | 2624 06 | 3258 5 | 3936 88 | 4655 62 | | | |
| $2\mathfrak{I}_c$ | 62 68 | 67 25 | 72 29 | 78.04 | 84.76 | 92 71 | 102.28 | 114.05 | 128.86 | 148.09 | 174.04 | 211.32 | – |
| $2\mathfrak{I}_{VMI}$ | 60 08 | 64 28 | 69.26 | 74.87 | 80 66 | 86 44 | 92.14 | 97.72 | 103.18 | 108.52 | | | |
| $(\hbar\omega)^2_c$ | 0 00712 | 0.0186 | 0 0329 | 0.0479 | 0.0618 | 0.073 | 0.0807 | 0.084 | 0.0826 | 0.0768 | 0.0669 | 0.054 | |
| $(\hbar\omega)^2_{VMI}$ | 0.0033 | 0 0125 | 0.0258 | 0.040 | 0.0559 | 0.0711 | 0.0862 | 0.1009 | 0.115 | 0.129 | | | |
| | | | | | | (b) | | | | | | | |
| $E_{exp}$ | 86.79 | 283.79 | 581.03 | 966.71 | 1428.59 | 1951.4 | 2514.9 | 3091.6 | 3672.1 | – | | | |
| $E_c$ | 84.63 | 282.13 | 580.8 | 968.98 | 1434.96 | 1967.06 | 2553.62 | 3182.93 | 3843.33 | 4523.12 | 5210.62 | 5894.15 | 6562 03 |
| $E_{VMI}$ | 86.77 | 284 36 | 583 33 | 973.13 | 1443 43 | 1986.04 | 2593.68 | 3260.42 | 3981.3 | – | – | – | – |
| $2\mathfrak{I}_c$ | 70.17 | 72.13 | 75 39 | 79 34 | 83 91 | 89.16 | 95.18 | 102.04 | 110.2 | 119.69 | 131.11 | 144.7 | 161 67 |
| $2\mathfrak{I}_{VMI}$ | 69.14 | 70.85 | 73.58 | 76.98 | 80.77 | 84.77 | 88.86 | 92.99 | 97.1 | – | – | – | – |
| $(\hbar\omega)^2_c$ | 0.0057 | 0.0162 | 0.0303 | 0 0464 | 0.063 | 0.079 | 0.0932 | 0.1049 | 0.113 | 0.1176 | 0.1182 | 0.1148 | 0.1076 |
| $(\hbar\omega)^2_{VMI}$ | 0 0025 | 0.0103 | 0.0228 | 0.0384 | 0.0557 | 0.07402 | 0.0926 | 0.1148 | 0.13023 | – | | | |

Table 3. The same as in Table 2 but for (a) $^{162}$Er [13–15] and (b) $^{164}$Er [14].

| $I$ | $2^+$ | $4^+$ | $6^+$ | $8^+$ | $10^+$ | $12^+$ | $14^+$ | $16^+$ | $18^+$ | $20^+$ | $22^+$ | $24^+$ | $26^+$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | (a) | | | | | | | |
| $E_{exp}$ | 102.08 | 329.63 | 666.76 | 1096.8 | 1602.9 | 2165.1 | 2745.7 | 3292.3 | 3846.5 | 4462.8 | - | - | - |
| $E_c$ | 100.6 | 328.37 | 666.15 | 1096.74 | 1602.99 | 2167.69 | 2773.69 | 3403.78 | 4040.8 | 4667.58 | 5266.92 | 5821.65 | - |
| $E_{VMI}$ | 100.19 | 326.71 | 660.93 | 1087.77 | 1594.39 | 2170.72 | 2809.00 | 3503 04 | 4247.83 | 5039.27 | - | - | - |
| $2\mathfrak{J}_c$ | 60.12 | 63.18 | 67.3 | 72.25 | 78.12 | 85.1 | 93 52 | 103.82 | 116.7 | 133.29 | 155.4 | 186 3 | - |
| $2\mathfrak{J}_{VMI}$ | 59.45 | 62.11 | 65.82 | 70.28 | 75.01 | 79.81 | 84.6 | 89.33 | 93.98 | 98.55 | - | - | - |
| $(\hbar\omega)^2_c$ | 0.0077 | 0.021 | 0.038 | 0.056 | 0.0728 | 0.0867 | 0 0965 | 0.1013 | 0.10073 | 0 0948 | 0.084 | 0.0693 | - |
| $(\hbar\omega)^2_{VMI}$ | 0.0034 | 0.01352 | 0.02861 | 0.0462 | 0.0647 | 0.08351 | 0.1023 | 0.1202 | 0.13901 | 0.1569 | - | - | - |
| | | | | | | (b) | | | | | | | |
| $E_{exp}$ | 91.39 | 299.47 | 614.34 | 1024.3 | 1517.8 | 2082.3 | 2700.8 | 3261.3 | 3766.8 | - | - | - | - |
| $E_c$ | 90.39 | 298.93 | 614.07 | 1024.25 | 1517 93 | 2083.53 | 3709.53 | 3289.35 | 4096.44 | 4834.26 | 5586.25 | 6340.84 | 7086.5 |
| $E_{VMI}$ | 90.99 | 298.04 | 610.96 | 1018.4 | 1509.81 | 2076.2 | 2710.03 | 3405.13 | 4156.33 | 4959.3 | - | - | - |
| $2\mathfrak{J}_c$ | 66.26 | 68.39 | 71.4 | 75.33 | 79.01 | 83.72 | 88.98 | 94.98 | 101.87 | 109.87 | 119.25 | 130.4 | 143.9 |
| $2\mathfrak{J}_{VMI}$ | 65.93 | 67.61 | 70.3 | 73.63 | 77.32 | 80.21 | 85.19 | 89.19 | 93.18 | 97.13 | - | - | - |
| $(\hbar\omega)^2_c$ | 0.0069 | 0.018 | 0.0338 | 0.052 | 0.071 | 0.0896 | 0.1066 | 0.1211 | 0.1322 | 0.1395 | 0.1426 | 0.1414 | 0.13359 |
| $(\hbar\omega)^2_{VMI}$ | 0.0027 | 0.01137 | 0.0251 | 0.04205 | 0.0609 | 0.0807 | 0.1008 | 0.12116 | 0.1414 | 0.1615 | - | - | - |

Table 4. The same as in Table 2 but for (a) $^{168}$Yb [16,17] and (b) $^{172}$Hf [18].

| $I$ | $2^+$ | $4^+$ | $6^+$ | $8^+$ | $10^+$ | $12^+$ | $14^+$ | $16^+$ | $18^+$ | $20^+$ | $22^+$ | $24^+$ | $26^+$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | (a) | | | | | | | |
| $E_{exp}$ | 87.73 | 286.55 | 585.3 | 970.05 | 1425.4 | 1935.9 | 2488.5 | 3073.00 | 3686.8 | 4336.8 | - | - | - |
| $E_c$ | 85 83 | 285.58 | 585 2 | 969.83 | 1425.83 | 1938.73 | 2494.3 | 3078.28 | 3676.41 | 4274.46 | 4858.16 | 5413.26 | - |
| $E_{VMI}$ | 86.9 | 284.13 | 580.64 | 965.26 | 1427.5 | 1958.6 | 2551.5 | 3200.4 | 3900 5 | 4647 7 | - | - | - |
| $2\mathfrak{J}_c$ | 69.17 | 71.6 | 75.6 | 80.53 | 86.39 | 93.28 | 101.48 | 111.32 | 123.4 | 138.4 | 157.5 | 183.02 | - |
| $2\mathfrak{J}_{VMI}$ | 69.04 | 71.02 | 74.17 | 78.22 | 82.21 | 86.61 | 91.07 | 95.54 | 99.99 | 104.38 | - | - | - |
| $(\hbar\omega)^2_c$ | 0.0059 | 0.0164 | 0.030 | 0.045 | 0.0595 | 0.0722 | 0.082 | 0.0881 | 0.0902 | 0.088 | 0.0817 | 0.0718 | - |
| $(\hbar\omega)^2_{VMI}$ | 0.0025 | 0.0103 | 0.0226 | 0.0375 | 0.0534 | 0.0709 | 0.0882 | 0.1056 | 0.1228 | 0.13986 | - | - | - |
| | | | | | | (b) | | | | | | | |
| $E_{exp}$ | 95.17 | 309.2 | 628.2 | 1037.2 | 1520.9 | 2064.3 | 2653.6 | 3276.7 | 3918.8 | - | - | - | - |
| $E_c$ | 94.67 | 309.04 | 628.18 | 1037.16 | 1521.05 | 2064.9 | 2653.85 | 3272.91 | 3907.17 | 4541.7 | 5161.58 | 5751.86 | - |
| $2\mathfrak{J}_c$ | 63.94 | 67.18 | 71.23 | 75.85 | 81.43 | 87.99 | 95.73 | 105.16 | 116.3 | 130.4 | 148.3 | 171.95 | - |
| $(\hbar\omega)^2_c$ | 0.0068 | 0.0187 | 0.043 | 0.0507 | 0.067 | 0.0811 | 0.921 | 0.099 | 0.1014 | 0.0991 | 0.0923 | 0.0813 | - |

Table 5. The same as in Table 2 but for (a) $^{174}$Hf [16,17] and (b) $^{176}$Hf [13].

| $I$ | $2^+$ | $4^+$ | $6^+$ | $8^+$ | $10^+$ | $12^+$ | $14^+$ | $16^+$ | $18^+$ | $20^+$ | $22^+$ | $24^+$ | $26^+$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | (a) | | | | | | | |
| $E_{exp}$ | 90.01 | 297.45 | 608.37 | 1009.42 | 1485.9 | 2020.6 | 2596.8 | - | - | - | - | - | - |
| $E_c$ | 88.65 | 296.08 | 607.82 | 1009.4 | 1486.34 | 2024.15 | 2608.37 | 3224.52 | 3858.1 | 4494.68 | 5119.74 | 5718.82 | - |
| $2\mathfrak{J}_c$ | 66.73 | 68.86 | 72.53 | 77.09 | 82.48 | 88.84 | 96.35 | 105.32 | 116.19 | 129.62 | 146.6 | 168.8 | - |
| $(\hbar\omega)^2_c$ | 0.0063 | 0.0177 | 0.0327 | 0.0492 | 0.0653 | 0.0796 | 0.0909 | 0.0985 | 0.1016 | 0.1002 | 0.0944 | 0.0844 | - |
| | | | | | | (b) | | | | | | | |
| $E_{exp}$ | 88.25 | 290.95 | 596.95 | 997.94 | 1481.27 | 2034.9 | 2646.8 | - | - | - | - | - | - |
| $E_c$ | 86.82 | 289.41 | 596.78 | 997.93 | 1481.86 | 2037.5 | 2654.07 | 3320.25 | 3908.8 | 4614.32 | 5333.76 | 6056.12 | 6770.41 |
| $2\mathfrak{J}_c$ | 68.45 | 70.21 | 73.11 | 76.59 | 80.58 | 85.1 | 90.23 | 96.06 | 102.73 | 110.41 | 119.37 | 129.93 | 142.56 |
| $(\hbar\omega)^2_c$ | 0.006 | 0.017 | 0.0322 | 0.0498 | 0.0684 | 0.0867 | 0.1036 | 0.1183 | 0.13 | 0.138 | 0.1423 | 0.1424 | 0.1384 |

similar steep increase can be predicted for $^{168}$Yb. The calculation from the CP formula shows that the increase in the moment of inertia is so rapid that the rotational frequency actually decreases as higher spin states are reached.

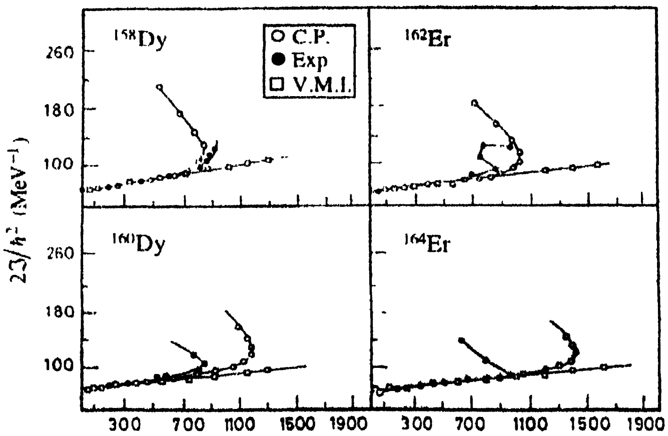


**Figure 1.** The observed moments of inertia as a function of the square of the angular velocity (backbending plot) showing experimental and theoretical curves (see text) for $^{158,160}$Dy and $^{162,164}$Er nuclei upto $I \simeq 26\hbar$. The figure shows the predicted value of the critical spin at which backbending is expected from the calculations.

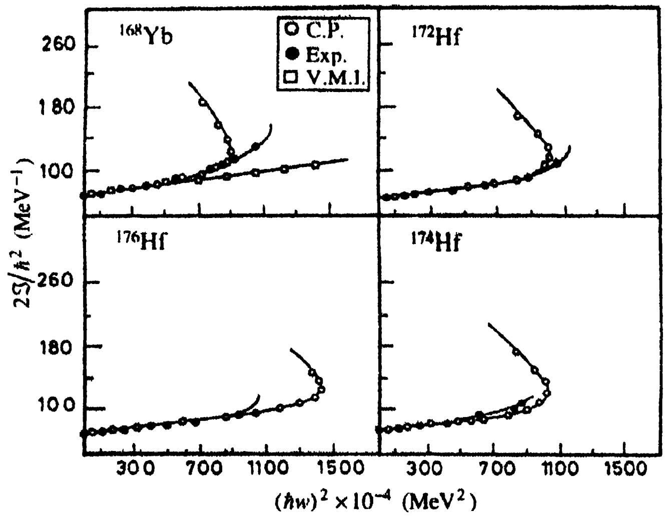


**Figure 2.** The same as in Figure 1 but for $^{168}$Yb and $^{172,174,176}$Hf nuclei.

This indicates that at a certain value of $I$, all these nuclei seem to exhibit the backbending behaviour. For high spin values, the first factor in eq. (2) is practically unity, and $(\hbar\omega)^2$ is seen to reach its maximum for the spin value

$$I_m = \frac{b}{3|c|}. \tag{4}$$

Since the spins of interest are only even integers, the even integer nearest to $I_m$ can be defined as the critical spin $I_c$ and this should satisfy the following condition :

$$2[\hbar\,\omega\,(I_c)]^2 < [\hbar\,\omega\,(I_c - 2)]^2.$$

The calculated $I_c$ values agree reasonably well with experimentally observed values given in Table 6.

**Table 6.** Predicted values of critical spin $I_c$ at which backbending is expected from the present calculation compared with the available experimental data [10,11] and with that in Ref. [12].

| Nucleus | $I_c$ (exp) | $I_c$ (cal.) | $I_c$ (Ref. [12]) |
|---|---|---|---|
| $^{158}$Dy | 12 | 16 | 16 |
| $^{160}$Dy | 14 | 22 | 24 |
| $^{162}$Er | 14 | 18 | 18 |
| $^{164}$Er | 14 | 22 | 24 |
| $^{168}$Yb | 12 | 22 | 22 |
| $^{172}$Hf | 16 | 18 | 20 |
| $^{174}$Hf | – | 18 | – |
| $^{176}$Hf | – | 22 | – |

## 4. Conclusion

A phenomenological description of the ground state band energies of the nuclei $^{158,160}$Dy, $^{162,164}$Er, $^{172,174,176}$Hf and $^{168}$Yb, based on an anharmonic vibrational picture, is developed. The proposed model with three parameter cubic polynomial (CP), gives excellent fits to the experimental spectra even up to $I = 26\hbar$. From the observed backbending behaviour, two general features are worth mentioning *viz.*, (i) the positions of the backbends can be reproduced correctly and (ii) the calculated backbends seem to be too sharp as compared with the experimental values. Also in the observed backbending behaviour, the rapid change in the moment of inertia can be interpreted in terms of a phase transition from a pair-correlated or superfluid state to a normal state without pair correlations due to coriolis anti-pairing effect. In this state, the nuclei are expected to rotate with the rigid body value of the moment of inertia.

### References

[1] A Johnson and Z Szymansky *Phys. Rep.* **7** 181 (1973)
[2] B R Mottelson and J G Valatin *Phys. Rev. Lett.* **5** 511 (1960)
[3] A Klein, R M Dreizler and T K Das *Phys. Lett.* **31B** 333 (1970)
[4] A N Mantri and P C Sood *J. Phys. Soc. Jpn. Suppl* **34** 456 (1973)
[5] T K Das and B Banerjee *Phys. Rev.* **C6** 2590 (1973)
[6] A N Mantri and P C Sood *Phys. Rev.* **C7** 1294 (1973)
[7] Y P Varshni *J. Phys. Soc. Jpn.* **36** 317 (1974)
[8] A N Mantri and P C Sood *Phys. Rev.* **C9** 2076 (1974)
[9] A Bohr and B R Mottelson *Nuclear Structure* **Vol. 2** p 80, p416, p580 (New York : Benjamin) (1975)
[10] R A Sorensen *Rev. Mod. Phys.* **45** 353 (1973)
[11] F S Stephens, P Kleinheinz, R K Sheline and R S Simon *Nucl Phys.* **A222** 235 (1974)
[12] P C Sood and A K Jain *Phys. Rev.* **C12** 1064 (1975)
[13] A Johnson, H Ryde and S A Hjorth *Nucl. Phys.* **A179** 753 (1972)

[14] R M Diamand and F S Stephens *Ann. Rev. Nucl. Part. Sci.* (1980) 30-85 and references there in

[15] G B Hagemann, R Broda, B Herskind, M I Shihara, S Ogazo and H Ryde *Nucl. Phys.* **A245** 166 (1975)

[16] K Schiffer, J D Garrett, G B Hagemann, B Hersking, B Lauritzen, R Chapina, D Clarke, F Khazaie, J C Lisle and J N Mo *Nuclear Structure*, Appendix to the Daresbury Annual Report p 57 (1987/88)

[17] J C B Acelar, C Ellegard, J D Garrett, G B Hagemann, B Herskind, A Holm, C-X Yang, P O TjØm and J C Lisle *Nuclear Structure*, Appendix to the Daresbury Annual Report p 39 (1983/84)

[18] R Chapman, J C Lisle, J N Mo, A Simcock, J C Willmott, J R Leslie, H G Price, P M Walker, J C Bacelar, J D Garrett, G B Hagemann, B Herskind, A Holm and P J Nolan *Phys. Rev. Lett* **51** 2265 (1983)